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TECHNICAL NOTE

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PRELIMINARY SURVEY OF RETROGRADE VELOCITIES REQUIRED FOR INSERTION INTO LOW-ALTITUDE LUNAR ORBITS

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Space Task Group
Langley Field, Va.

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PRELIMINARY SURVEY OF RETROGRADE VELOCITIES REQUIRED FOR INSERTION INTO LOW-ALTITUDE LUNAR ORBITS

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SUMMARY

Closed lunar orbits are envisaged in lunar mission programs. The study described herein was undertaken to obtain an appreciation of the relevant fuel consumption requirements. The retrograde impulses necessary for establishing the orbits were assumed to occur at the point of closest approach of the main earth-moon trajectory; this point, designated as the arrival position, was restricted to a lunar altitude of 5,000 nautical miles or less. The orientation of the arrival position vector relevant to any coplanar radius vector is not constrained, however, and similarly the scalar value of the arrival velocity is unrestrained.

Since the arrival altitude is restricted to 5,000 nautical miles or less, the perturbing accelerations of the earth and sun are sufficiently small that the vehicle and moon essentially comprise an isolated two-body system; this is discussed in the report.

Retrograde velocities are determined for any required pericynthion position. If the pericynthion orientation requirement is relaxed then a smaller retrograde velocity is in some cases possible. A comparison between minimum retrograde velocities and retrograde velocities necessary for stipulated pericynthion positions is given. Arrival velocities are correlated with feasible earth departure conditions.

The equations developed for determining retrograde velocities for desired pericynthion positions are considered useful for estimating essential data for the preliminary planning of lunar missions. Some graphical representation is included herein for immediate familiarization with possible conditions.

INTRODUCTION

The study described in this report was initiated by the desire for a preliminary assessment of various problems associated with local lunar orbits as envisaged in lunar mission programs. Since it was known that the lunar gravitational field would be dominant during low-altitude orbits and that two-body solutions would be applicable, it was decided to use a closed-solution approach. When satellites of a dominating gravitational field are studied, velocities relative to the nonrotating coordinate system, with origin at the gravity source, are treated as inertial in order that Newtonian laws may apply. As an example, planet velocities relative to the sun are considered inertial and yet the sun is thought to be moving in space. For a lunar satellite, a perfectly elliptical orbit cannot be achieved due to the movements of the sun, earth, and moon relative to the vehicle; however, for low-altitude lunar orbits, the lunar gravitational field is dominant; hence, near elliptical orbits may occur. Stability checks of the ellipticity were examined from the output of a digital program employing the Encke technique of integration with origin at the moon center. This was considered to be sufficiently accurate and the checks confirmed the validity of the approach taken.

Contained within this report is a method for determining the retrograde velocity necessary for establishing orbits with specific characteristics. It is anticipated that the plane of the vehicle's lunar arrival velocity will, by guidance impulses, nearly coincide with the plane of the desired resultant orbit; hence, one constraint of this study was that both the arrival velocity and resultant orbit be coplanar. This report also indicates the correlation of arrival velocity with specified earth departure conditions by means of a restricted three-body mathematical model.

It is possible in many cases to reduce the magnitude of the retrograde impulse if the restrictions on the pericynthion orientation are relaxed. Therefore, a method is presented whereby the minimum retrograde impulse can be determined for an orbit with a specified pericynthion radius with no constraint on orientation. The methods of determining minimum retrograde velocities and retrograde velocities for required response are thought to be useful in that they point the way to a comprehensive quantitative survey program for assessing fuel requirements for lunar orbits.

SYMBOLS

a	semimajor axis of orbit, ft
C	Jacobian constant, ft^2/sec^2
D	distance from center of earth to center of moon, ft

e	Naperian logarithm base
G	universal gravitational constant, $ft^4\text{-}lb/sec^4$
g_o	gravitational acceleration at earth's surface, ft/sec^2
h	altitude, nautical miles
h_a	altitude at apocynthion, nautical miles
h_p	altitude at pericynthion, nautical miles
I	specific impulse of fuel, sec
M	mass of earth plus mass of moon, slugs
M_e	mass of earth, slugs
M_m	mass of moon, slugs
m	mass of lunar vehicle, slugs
r	distance from moon center to vehicle position, ft
r_a	apocynthion distance from moon center, ft
r_{bp}	distance from barycenter to vehicle position, ft
r_{ep}	distance from earth center to vehicle position, ft
r_p	pericynthion distance from moon center, ft
R_e	distance from earth center to barycenter, ft
R_m	distance from moon center to barycenter, ft
t	time, sec
V	velocity referred to the rotating coordinate system, origin the barycenter, ft/sec
V_A	arrival velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_e velocity referred to the nonrotating coordinate system, origin the earth center, ft/sec

V_i velocity referred to the nonrotating coordinate system, origin the barycenter, ft/sec

V_m velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_o orbit velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_{oa} orbit apocynthion velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_{op} orbit pericynthion velocity referred to the nonrotating coordinate system, origin the moon center, ft/sec

V_R retrograde velocity, ft/sec

X,Y,Z rotating coordinates, origin the barycenter

X_e,Y_e,Z_e nonrotating coordinates, origin the earth center

X_i,Y_i,Z_i nonrotating coordinates, origin the barycenter

X_m,Y_m,Z_m nonrotating coordinates, origin the moon center, X_m and Y_m in the plane of the vehicle's orbit

γ angle between vehicle velocity vector and local horizontal, deg

δ ratio of fuel mass to gross mass at commencement of thrusting for entry into local lunar orbit

$\bar{\delta}$ overall ratio of fuel mass to gross mass required for both entry into and exit from the local lunar orbit at the same point of orbit

ϵ eccentricity

θ angle between r and r_p at instant of retroimpulse burnout, deg

$\mu = \frac{M_m}{M}$

$1 - \mu = \dfrac{M_e}{M}$

ζ included angle between V_A and V_R, deg

ω rotational velocity of earth-moon system, radians/sec

A dot over a symbol indicates differentiation with respect to time.

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DISCUSSION

Arrival Velocity

In lunar mission programs, it is assumed that one of the mission's main objectives will be to make a close survey of the moon's terrain. For a detailed survey, this requires the vehicle to establish an orbit about the moon which will necessitate the application of a retrograde impulse. It is assumed for this study that the impulse will be applied at the instant of closest approach to the moon when the vehicle's velocity vector is normal to the extension of the moon's radius. A conception of the complete mission is shown in figure 1.

The vehicle's lunar arrival velocity and position can be correlated with the earth insertion velocity and position by reference to a restricted three-body mathematical model. In this model the moon and earth are considered to rotate with constant radii and constant angular velocity about their common center of mass. The earth and moon are considered as point masses and the vehicle's mass is regarded as infinitely small in comparison. Further information concerning the characteristics of the restricted three-body mathematical model is found in reference 1.

The restricted three-body equations of motion as given in the rotating coordinate system with the barycenter as origin are

$$\ddot{X} = \omega^2 X + 2\omega\dot{Y} - \frac{GM(1 - \mu)\left(X - X_1\right)}{r_{ep}^3} - \frac{GM\mu\left(X - X_2\right)}{r^3} \tag{1}$$

$$\ddot{Y} = \omega^2 Y - 2\omega\dot{X} - \frac{GM(1 - \mu)Y}{r_{ep}^3} - \frac{GM\mu Y}{r^3} \tag{2}$$

$$\ddot{Z} = \frac{-GM(1 - \mu)Z}{r_{ep}^3} - \frac{GM\mu Z}{r^3} \tag{3}$$

where $\left(X - X_1\right)$ is the distance along the X coordinate from the earth center to the particle and $\left(X - X_2\right)$ is the distance along this coordinate from the moon center to the particle.

By writing

$$W(X,Y,Z) = \frac{1}{2}\,\omega^2\left(X^2 + Y^2\right) + \frac{GM(1 - \mu)}{r_{ep}} + \frac{GM\mu}{r} \tag{4}$$

then the equations of motion become

$$\ddot{X} = \frac{\partial W}{\partial X} + 2\omega\dot{Y} \tag{5}$$

$$\ddot{Y} = \frac{\partial W}{\partial Y} - 2\omega\dot{X} \tag{6}$$

$$\ddot{Z} = \frac{\partial W}{\partial Z} \tag{7}$$

By multiplying equations (5) to (7) by $2\dot{X}$, $2\dot{Y}$, and $2\dot{Z}$, respectively, adding, and integrating, Jacobi's integral is obtained as

$$\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 = \omega^2\left(X^2 + Y^2\right) + \frac{2GM(1 - \mu)}{r_{ep}} + \frac{2GM\mu}{r} - C \tag{8}$$

or

$$C = \omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}} + \frac{2\mu GM}{r} - V^2 \tag{9}$$

where r_{bp} is in the earth-moon plane.

If the earth departure velocity and position are known, the integration constant C can be determined. Once C is determined, it is possible to calculate the scalar value of the lunar arrival velocity for any given arrival position. The transformation of the velocity in the rotating coordinate system, origin the barycenter, to the velocity in the nonrotating coordinate system, origin the moon center, is

$$\vec{V}_m = \vec{V} + \left(\vec{\omega} \times \vec{r}\right) \tag{10}$$

The relationship between the velocities in the different coordinate systems is illustrated in figure 2. A comparison between earth departure velocities and lunar arrival velocities is shown in figure 3. The correlation between arrival velocity and earth departure velocity is shown for the condition of minimum earth-moon distance. The vehicle's arrival velocity is in the $\left(-\vec{\omega} \times \vec{r}\right)$ direction.

In acquiring a quantitative feel for lunar arrival velocities as a function of earth departure velocities for a given earth departure position, it is natural to think in terms of V_e rather than V. A knowledge of the value of V does not immediately give the value of V_e; however, for the departure position given in figure 3, it is known that γ will be in the range of -2° to 25° in the case of coplanar trajectories. Consequently, if the rotational rate of the earth-moon system and the distance between the earth center and barycenter are known, then, for a given value of V, the associated range of V_e may be determined. Through the Jacobian relationship, the corresponding value of V at the arrival position, point of closest approach to the moon, may be established. Since V at the arrival position is perpendicular to the polar position vector from the moon center, V_A referred to a nonrotating axis system (origin the moon center) may be immediately determined.

It may be shown that for a given value of C and arrival position vector, regardless of the orientation of the position vector, the lunar arrival velocity is approximately constant. This assertion must be qualified by restricting the altitude to the range considered in this study. The justification for the approximation is given in appendix B.

Determination of Retrograde Velocities

This section contains a method for determining instantaneous retrograde velocities within certain constrained conditions. For the method described herein, the plane of action is defined as the plane containing the moon center and the arrival velocity vector. The retrograde impulses will be applied in this plane and, therefore, the resulting orbits will be in this plane. Also, since the retrograde impulse is to be initiated at the point of closest approach to the moon, then the arrival velocity vector will be normal to the position vector.

The method for determining an orbit with a required pericynthion radius and orientation is now presented; the two-body relationships are developed in appendix A. The following sketch depicts the vehicle's arrival at the vicinity of the moon where the plane of arrival is denoted as the X_mY_m plane:

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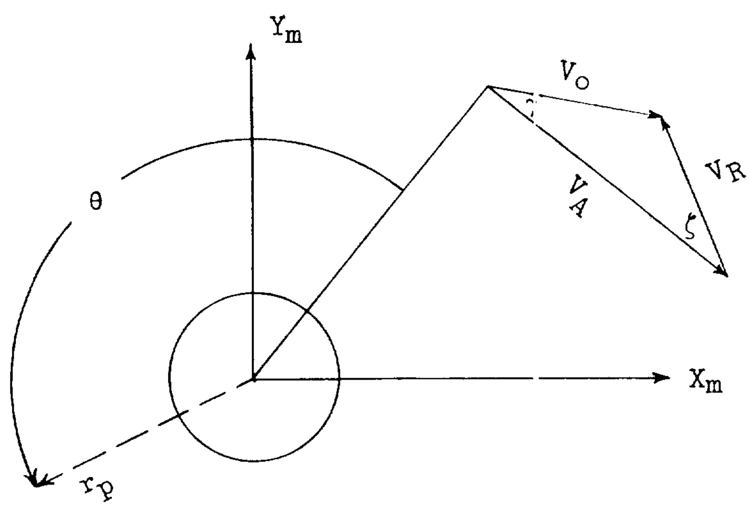

where $\theta \leqq 180$, measured clockwise or counterclockwise from the r vector. By the law of cosines

$$V_R^2 = V_A^2 + V_o^2 - 2V_oV_A \cos \gamma \tag{11}$$

where

$$\cos \gamma = \frac{r_pV_{op}}{rV_o} \tag{12}$$

and

$$V_o^2 = GM_m\left(\frac{2}{r} - \frac{1}{a}\right) \tag{13}$$

Substitution of equations (12) and (13) into equation (11) yields

$$V_R^2 = \frac{GM_m(2a - r)}{ar} + V_A^2 - \frac{2V_A r_p V_{op}}{r} \tag{14}$$

It is seen from the relationships in appendix A that

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$$a = \frac{r_p\left(\frac{r_p}{r} - \cos\theta\right)}{2\frac{r_p}{r} - 1 - \cos\theta} \tag{15}$$

It also follows from the relationships in appendix A that

$$V_{op} = \left[GM_m \cdot \frac{(1 - \cos\theta)}{r_p\left(\frac{r_p}{r} - \cos\theta\right)}\right]^{1/2} \tag{16}$$

Substitution of equation (15) into equation (13) yields

$$V_o^2 = \frac{GM_m}{r_p}\left[\frac{r}{r_p}\left(\frac{1 - \cos\theta}{1 - \frac{r}{r_p}\cos\theta}\right) + 2\left(\frac{r_p}{r} - 1\right)\right] \tag{17}$$

Now, let

$$\frac{r_p}{r} \equiv \phi \tag{18a}$$

$$\frac{GM_m}{\phi r_p} \equiv A^2 \tag{18b}$$

$$\frac{1 - \cos\theta}{1 - \frac{1}{\phi}\cos\theta} \equiv B \tag{18c}$$

$$\frac{2GM_m}{r_p}(\phi - 1) \equiv K \tag{18d}$$

Substitution of identities (18a) to (18d) into equation (17) yields the following equation:

$$V_o^2 = A^2B + K \tag{19}$$

Equation (11) may now be written as

$$V_R^2 = V_A^2 + A^2B + K - 2V_A\phi AB^{1/2} \tag{20}$$

Equation (20) is the general equation which determines the retrograde velocity for the required conditions.

Transforming equation (12) yields

$$\gamma = \pm \text{arc cos}\left(\frac{r_p}{r}\right)\left(\frac{V_{op}}{V_o}\right) \tag{21}$$

By determining γ then ζ can be calculated from the law of sines. Thus,

$$\zeta = \text{arc sin}\left(\frac{V_o}{V_R} \sin \gamma\right) \tag{22}$$

Equations (20) and (22) determine the required retrograde velocity vector. However, it should be noted that the resultant trajectory could represent any type of conic orbit depending on the arrival conditions and the required characteristics of the resulting trajectory. It is assumed that an elliptical orbit is desired; however, it does not necessarily follow that the retrograde impulse will yield an elliptical orbit. The classification of the conic orbit can be found by determining the eccentricity where for an elliptical orbit $0 < \epsilon < 1$, for a parabolic orbit $\epsilon = 1$, and for a hyperbolic orbit $\epsilon > 1$.

Although it is not envisaged that it will ever be desirable to arrive at a certain pericynthion position on a hyperbolic trajectory, it is possible to determine the necessary retrograde impulse vector which will accomplish this. If equation (20) is modified to

$$V_R^2 = V_A^2 + A^2B + K + 2V_A\phi AB^{1/2} \tag{23}$$

then solution of this equation (23) will yield the necessary velocity. The orbit velocity vector in this case will be opposite in direction to that which would be determined if equation (20) were utilized.

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In order to obtain an insight into the quantitative values of retrograde velocity, refer to figure 4 and table 1. Figure 4 shows the retrograde velocities necessary for various combinations of arrival and required conditions and the orbit reference table defines the orbits established as a result of these retrograde impulses. As an example, suppose it is desired to establish an orbit with a pericynthion altitude of 100 nautical miles with the pericynthion radius oriented 150° from the insertion radius. Assume that the vehicle arrives at an altitude of 1,000 nautical miles with a velocity of 6,500 ft/sec. Reference to figure 4 shows that this will require a retrograde impulse of 3,360 ft/sec. Reference to table 1 shows that the established orbit will have an apocynthion altitude of 1,128.6 nautical miles, an apocynthion velocity of 3,031.5 ft/sec, a pericynthion velocity of 6,036.9 ft/sec, and the time required for one complete orbit will be 3.8 hours.

It has been determined that the preceding equations will yield the retrograde impulse for any specified values of pericynthion radius and orientation. However, it is possible in many cases to reach these pericynthion altitudes with smaller retrograde velocities if the restrictions on the orientation of pericynthion are relaxed. The determination of minimum retrograde velocity for a given scalar value of pericynthion altitude is given in appendix C.

A comparison between minimum retrograde velocities and retrograde velocities for constrained conditions is shown in figure 5. For the construction of this figure, the vehicle is assumed to arrive at an altitude of 1,000 nautical miles with an arrival velocity of 7,000 ft/sec. Figure 5(a) shows r_p/r, where r is the arrival radius, plotted against θ which will yield the minimum retrograde velocity. Figure 5(b) shows r_p/r plotted against the required minimum retrograde velocity and the comparison (dashed) curve shows the retrograde velocities which yield a pericynthion orientation of 150°. Figure 5(c) shows r_p/r plotted against the resultant orbit apocynthion radius for the case where the retrograde impulse is a minimum and for the comparison case. It can be seen that over a large range of r_p/r, the minimum retrograde impulse will offer some fuel savings without influencing the resultant orbits significantly. In the range of r_p/r where the fuel savings are considerable, however, the resultant orbits have the following undesirable features: large apocynthion radii, large pericynthion velocities, and large orbit periods.

In order to improve the quantitative feel for retrograde velocities as a function of free-coast earth-moon trajectories, retrograde velocities for $\theta = 180^{\circ}$ are plotted against arrival velocity and altitude for a fixed earth-moon distance and earth departure position vector in figure 6; in sequence (figs. 6(a) to 6(d)) the value of the departure velocity V is varied. On the assumption that the departure angle γ is between -2° and 25° the normally referred to earth departure velocity V_e may be quickly determined within the limits of $\pm 3\frac{1}{2}$ ft/sec and in some cases direct reference may be made to figure 3.

Fuel Requirements

The fuel required for instantaneous retrograde impulse for establishing lunar orbits can be calculated from the equation

$$\delta = 1 - e^{-V_R/Ig_o} \tag{24}$$

where δ is the ratio of the fuel mass at commencement of burning to the gross mass. Shown in figure 7 are the ratios of fuel mass to gross mass necessary for establishing the orbits shown in figure 6. The fuel-consumption values as shown are considered to be absolute minimums since the impulses are assumed to be instantaneous. Fuel-consumption values for corresponding finite impulses, however, will differ little from the given values. In all cases where thrust is acting against a resolved weight component, there is a loss in efficiency, and this to a small degree would be the case with corresponding impulses of finite duration.

As an example of the use of this graph, consider a lunar vehicle with an earth-surface weight of 10,000 pounds containing a fuel of 250 seconds specific impulse. From the graph it will be noted that, within the conditions considered, the earth-surface weight of fuel required for inserting the vehicle at a lunar altitude of 5,000 nautical miles into an orbit with a pericynthion altitude of 100 nautical miles is 3,200 pounds.

In order to obtain the total fuel requirement for orbit entry and exit, the following considerations are taken into account. The retrograde velocity required to insert into the orbit is assumed to equal the posigrade velocity for exit. This assumption is made since the velocity requirements relative to the center of the moon of the major earth-moon trajectory will not have substantially changed after a restricted number of lunar orbits. The total fuel requirement is found from the equation

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$$\bar{\delta} = 1 - e^{-2V_R/Ig_0} \tag{25}$$

or

$$\bar{\delta} = 2\delta - \delta^2 \tag{26}$$

where $\bar{\delta}$ is the ratio of total fuel mass to gross mass required for orbit entry and exit. Shown in figure 8 are the ratios of total fuel mass to gross mass necessary for orbit entry and exit for the orbits shown in figure 6. As an example of the use of figure 8, consider a lunar vehicle with an earth-surface weight of 10,000 pounds containing a fuel of 250 seconds specific impulse. From figure 8(c) it can be deduced that the earth-surface weight of fuel required for both entry and exit, at an arrival altitude of 5,000 nautical miles for an orbit with a pericynthion altitude of 100 nautical miles, is approximately 5,400 pounds.

Orbit Stability

The simplest and possibly the best method of considering the stability of the vehicle's orbit about the moon is to consider the vehicle as a satellite of the moon where the following effects are considered to be the major perturbation effects:

(1) the earth's gravity field

(2) the sun's gravity field

(3) the moon's potential distribution

(4) the lunar librations

By stability it is implied that successive orbits have repetitive characteristics.

A brief analysis of the individual gravitational effects of the earth, sun, and moon show the possibility of a highly stable orbit. The order of magnitude of the gravitational effects of the earth, sun, and moon on the vehicle are shown in the following table:

	Gravitational effects, expressed in ft/sec^2, for -	
	Zero lunar altitude	Lunar altitude of 5,000 nautical miles
Moon	5.31	0.13
Sun	.02	.02
Earth	.01	.01

If for a short duration there is insignificant difference between the effects on the moon and the effects on the vehicle due to the gravity fields of the earth and sun, then the moon and vehicle will tend to behave as a two-body system with a resultant stable orbit. Should the apocynthion altitude of the vehicle's orbit never be greater than 5,000 nautical miles, then the scalar accelerations of the vehicle and moon due to the sun never have a greater ratio than 1.000147 or the inverse. The directional difference in the acceleration vectors is negligible since the sun is approximately 80,764,000 nautical miles away. The corresponding ratio due to the earth is never greater than 1.0647 or the inverse and the directional difference is small. Viewed in this manner, the sun's gravitational effect is very small. The moon and vehicle acceleration vectorial difference due to the earth appears to be more significant, but due to the oscillatory nature of the difference (since it is periodic with the vehicle's orbit), the effect is small over a restricted number of orbits.

Other perturbation effects of interest are the librations of the moon about its center of gravity. The apparent librations as viewed from earth are of no concern in this study. There is a small real libration in longitude due to the eccentricity of the moon's orbit about the barycenter but the period is a month, and hence this libration does not present a problem. If any high-frequency librations exist, they are thought to be insignificantly small.

The main changes in orbit characteristics with time are anticipated to be changes in the inclination of the orbit and regression of the nodes relative to the lunar equator.

Several stability checks have been conducted by a comprehensive simulation incorporated in a digital mathematical model which includes earth, sun, and lunar potential distribution effects.

This model is particularly attractive in that the origin of integration is the moon for the conditions considered and the round-off errors are those involved in the integration of perturbations from reference ellipses. The checks added confidence to the preceding discussion of stability.

CONCLUSIONS

In a preliminary study of retrograde velocities required for insertion into lunar low-altitude orbits, the following conclusions were reached:

1. At low lunar altitudes, say less than 5,000 nautical miles, the lunar vehicle in free-coast conditions will essentially behave as a satellite of the moon and, hence, the trajectory will be conical with the center of the moon mass as a focal point. Depending on the velocity imparted by the retrograde impulse, the trajectory will be elliptic, parabolic, or hyperbolic. Elliptical trajectories are of interest in that closed orbits about the moon are required for survey purposes.

2. A simple thrusting logic may be introduced for obtaining a required pericynthion position, the only requirement being that the insertion velocity is that which would be yielded by the classical two-body solution for the required conditions.

3. The lunar arrival velocity increases with increase in earth insertion velocity and this entails heavier fuel expenditure for insertion into a local lunar orbit; however, there is the possibility that the trajectory associated with a higher earth insertion velocity will require less fuel expenditure for guidance before the arrival phase.

4. For a given arrival altitude, the nearer the required orbit is to a circular orbit, the lower the required retrograde velocity. Furthermore for a given pericynthion altitude, the nearer the associated orbit is to circular, the smaller the pericynthion velocity.

5. For an elliptical orbit with pericynthion altitude and orientation stipulated, there is a unique retrograde velocity. If the restriction on the orientation of the pericynthion is relaxed, then in many cases it is possible to determine a smaller retrograde impulse which will yield the desired pericynthion altitude.

6. There are cases when the resultant pericynthion of a minimum retrograde impulse does not occur directly opposite the point of application. Utilization of a minimum retrograde impulse in these circumstances may produce an orbit with an excessive apocynthion altitude, pericynthion velocity, and period. It is possible to minimize these adverse features, at the expense of fuel, by orienting the pericynthion away from the point of minimum retrograde impulse to a point where the orbit characteristics become more desirable.

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National Aeronautics and Space Administration, 1
Langley Field, Va., June 12, 1961. 0

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APPENDIX A

TWO-BODY EQUATIONS

In order to maintain continuity in the body of the text, transposes of well-known two-body relationships are immediately used. This appendix is included to indicate the derivation of these relationships.

The differential equations of motion are

$$m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right] = -\frac{GM_m m}{r^2} \tag{A1}$$

and

$$\frac{m}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0 \tag{A2}$$

From these equations and a knowledge of conic geometry, the following equations can be evolved. The general conic expression for the semilatus rectum is

$$p = r(1 + \epsilon \cos \theta) \tag{A3}$$

and when $r = r_p$, then $\theta = 0$ which yields

$$p = r_p(1 + \epsilon) \tag{A4}$$

The expression for the angle γ which is the angle between the velocity vector V_o and the normal to the vector r is from integration of equation (A2)

$$\cos \gamma = \frac{r_p V_{op}}{r V_o} \tag{A5}$$

The expression for the semimajor axis which is evolved from equation (A3) is

$$a = \frac{p}{1 - \epsilon^2} \tag{A6}$$

Substitution of the values of p and ϵ from equations (A3) and (A4) yield

$$a = \frac{r_p\left(\frac{r_p}{r} - \cos\theta\right)}{2\frac{r_p}{r} - 1 - \cos\theta} \tag{A7}$$

For a hyperbola since $\epsilon > 1$ it can be seen from equation (A6) that the numerical value of a will be negative. The orbit velocity is

$$V_o{}^2 = GM_m\left(\frac{2}{r} - \frac{1}{a}\right) \tag{A8}$$

Substituting equation (A7) into equation (A8) yields

$$V_o{}^2 = \frac{GM_m}{r_p}\left[\frac{r}{r_p}\left(\frac{1 - \cos\theta}{1 - \frac{r}{r_p}\cos\theta}\right) + 2\left(\frac{r_p}{r} - 1\right)\right] \tag{A9}$$

The pericynthion velocity is

$$V_{op}{}^2 = GM_m\left(\frac{2}{r_p} - \frac{1}{a}\right) \tag{A10}$$

Substitution for a from equation (A7) in equation (A10) yields

$$V_{op}{}^2 = GM_m\left[\frac{1 - \cos\theta}{r_p\left(\frac{r_p}{r} - \cos\theta\right)}\right] \tag{A11}$$

APPENDIX B

ARRIVAL VELOCITY APPROXIMATIONS

This appendix is presented to show that for a given value of C the arrival velocity is almost entirely dependent on lunar altitude, within the limits of this study. This is succeeded by a derivation of an approximation of the variation of arrival velocity with lunar altitude.

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Arrival Velocity at Constant Lunar Altitude

The Jacobian velocity relationship is as follows:

$$V^2 = \omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}} + \frac{2\mu GM}{r} - C \tag{B1}$$

For convenience, equation (B1) is rewritten so that the terms independent of the orientation of $\vec{r}$ appear on the left-hand side as

$$V^2 - \frac{2\mu GM}{r} + C = \omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}} \tag{B2}$$

Upon utilizing the law of cosines, the following equation is evolved:

$$\left[\omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}}\right] = \omega^2 \left[\left(R_m^2 + r^2\right) + 2R_m\left(X - R_m\right)\right] + \frac{2(1 - \mu)GM}{\left\{\left[\left(R_e + R_m\right)^2 + r^2\right] + 2\left(R_e + R_m\right)\left(X - R_m\right)\right\}^{1/2}} \tag{B3}$$

Since

$$\left.\begin{aligned}
&\left(R_m^2 + r^2\right) >>> 2R_m\left(X - R_m\right)\\
&\text{and}\\
&\left[\left(R_m + R_e\right)^2 + r^2\right] >>> 2\left(R_m + R_e\right)\left(X - R_m\right)
\end{aligned}\right\} \quad R_m >>> r; \; \left|X - R_m\right| \leqq r$$

Therefore,

$$\left[\omega^2 r_{bp}^2 + \frac{2(1 - \mu)GM}{r_{ep}}\right] \approx \omega^2\left(R_m^2 + r^2\right) + \frac{2(1 - \mu)GM}{\left[\left(R_m + R_e\right)^2 + r^2\right]^{1/2}}$$

This allows equation (B1) to be modified to a close approximation for the range of altitudes studied as follows:

$$V^2 \approx \omega^2\left(R_m^2 + r^2\right) + \frac{2(1 - \mu)GM}{\left[\left(R_m + R_e\right)^2 + r^2\right]^{1/2}} + \frac{2\mu GM}{r} - C \tag{B4}$$

Consequently, it can be seen from approximation (B4) that if r is constant, then the orientation of r will not affect the value of V, $\vec{V}_m = \vec{V} + \left(\vec{\omega} \times \vec{r}\right)$ where $\vec{\omega}$ is a constant vector and $\vec{r}$ has a constant scalar value. If trajectories in the earth-moon plane are being considered, then $V_m = V \pm \omega r$ where ωr is a constant.

Variation of Arrival Velocity With Lunar Altitude

The trend of the arrival velocity with increase of altitude above the lunar surface may be more clearly understood by the following considerations. Substitution of equation (B3) into equation (B1) yields

$$V^2 = \omega^2\left[R_m^{\,2} + r^2 + 2R_m\left(X - R_m\right)\right] + \frac{2(1 - \mu)GM}{\left[\left(R_e + R_m\right)^2 + 2\left(R_e + R_m\right)\left(X - R_m\right)\right]^{1/2}}$$

$$+ \frac{2\mu GM}{r} - C \tag{B5}$$

Differentiation of this equation with respect to r yields

$$\frac{\partial V^2}{\partial r} = 2\omega^2 r - \frac{2(1 - \mu)(GM)r}{\left[\left(R_e + R_m\right)^2 + r^2 + 2\left(R_e + R_m\right)\left(X - R_m\right)\right]^{3/2}} - \frac{2\mu GM}{r^2} \tag{B6}$$

however,

$$\left\{2\omega^2 r - \frac{2(1 - \mu)(GM)r}{\left[\left(R_e + R_m\right)^2 + r^2 + 2\left(R_e + R_m\right)\left(X - R_m\right)\right]^{3/2}}\right\} \lll \frac{2\mu GM}{r^2}$$

Therefore, it is possible to reduce equation (B6) to the approximation

$$\frac{\partial V^2}{\partial r} \approx \frac{-3.5 \times 10^{14}}{r^2}\ \mathrm{ft/sec^2} \tag{B7}$$

This expression gives a good approximation for the trend of the arrival velocity with altitude. Although this approximation applies for a velocity V, which is referred to the rotating axis system with origin at the barycenter, it is now shown that with little loss of accuracy V may be regarded as V_A. Given that

$$V_A^{\,2} = V^2 \pm 2\omega rV + \omega^2 r^2 \tag{B8}$$

then,

$$\frac{\partial V_A^{\,2}}{\partial r} = \frac{\partial V^2}{\partial r} + \frac{\partial}{\partial r}\left(\omega^2 r^2\right) \pm \frac{\partial}{\partial r}(2\omega r V) \qquad \text{(B9)}$$

which yields

$$\frac{\partial V_A^{\,2}}{\partial r} = \frac{\partial V^2}{\partial r} \pm \frac{\omega r}{V}\frac{\partial V^2}{\partial r} + 2\omega^2 r \pm 2\omega V \qquad \text{(B10)}$$

and upon collecting terms

$$\frac{\partial V_A^{\,2}}{\partial r} = \frac{\partial V^2}{\partial r}\left(1 \pm \frac{\omega r}{V}\right) + 2\omega(\omega r \pm V) \qquad \text{(B11)}$$

By inserting a relevant range of values into the equation it is found that

$$\frac{\omega r}{V} \lll 1$$

and

$$\left|2\omega(\omega r \pm V)\right| \lll \left|\frac{\partial V^2}{\partial r}\right|$$

Therefore, the following approximation is acceptable:

$$\frac{\partial V_A^{\,2}}{\partial r} \approx \frac{\partial V^2}{\partial r} \approx \frac{-3.5 \times 10^{14}}{r^2}\ \text{ft/sec}^2 \qquad \text{(B12)}$$

APPENDIX C

DERIVATION OF THE MINIMUM RETROGRADE VELOCITY TO OBTAIN A GIVEN PERICYNTHION SCALAR VALUE

Upon referring to the following equation

$$V_R{}^2 = V_A{}^2 + A^2B + K - 2V_A\phi AB^{1/2} \tag{C1}$$

and the identities (18a), (18b), and (18d), it is seen that the quantities A, K, and ϕ are independent of θ; therefore, differentiation of equation (C1) with respect to θ yields

$$\frac{\partial V_R{}^2}{\partial\theta} = A\left(A - V_A\phi B^{-1/2}\right)\frac{\partial B}{\partial\theta} \tag{C2}$$

and differentiation of identity (18c) yields

$$\frac{\partial B}{\partial\theta} = \frac{\left(1 - \frac{1}{\phi}\right)\sin\theta}{\left(1 - \frac{\cos\theta}{\phi}\right)^2} \tag{C3}$$

A necessary condition for minimum retrograde velocity is that $\frac{\partial V_R{}^2}{\partial\theta} = 0$, and it is known that $A \neq 0$; therefore,

$$\left(A - V_A\phi B^{-1/2}\right)\left[\frac{\left(1 - \frac{1}{\phi}\right)\sin\theta}{\left(1 - \frac{\cos\theta}{\phi}\right)^2}\right] = 0 \tag{C4}$$

Suppose, tentatively, that for equation (C4) to hold,

$$A - V_A \phi B^{-1/2} = 0 \tag{C5}$$

therefore,

$$B^{-1} = \frac{A^2}{\left(V_A \phi\right)^2} \tag{C5a}$$

Substitution of equation (C5a) into identity (18c) yields

$$\cos\theta = \frac{1 - \left(\frac{A}{V_A \phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A \phi}\right)^2} \tag{C6}$$

Equation (C4) has two boundary conditions $\theta = 0^\circ$ and $\theta = 180^\circ$. When $r = r_p$, then $\theta = 0$ by definition of the two-body orbit and this is the unique value of θ to be considered. By reference to equation (C4), it is seen that $\theta = 180^\circ$ is an alternative solution to

$$\frac{\partial V_R^{\,2}}{\partial\theta} = \frac{\left(1 - \frac{1}{\phi}\right)\sin\theta}{\left(1 - \frac{\cos\theta}{\phi}\right)^2} = 0 \tag{C7}$$

Two conditions are now stipulated. For condition 1 where $\theta = 180^\circ$ and $\frac{\partial V_R^{\,2}}{\partial\theta} = 0$ let $V_R = V_{R1}$. For condition 2 where $180 \geqq |\theta| \geqq 0$ and $\frac{\partial V_R^{\,2}}{\partial\theta} = 0$ let $V_R = V_{R2}$.

The following analysis is to prove that when V_{R2} exists, it is a minimum. Tentatively assume that $V_{R1}^{\,2} \geqq V_{R2}^{\,2}$ in which case the following equation is evolved from equation (C1):

$$A^2B_1 - 2V_A\phi AB_1{}^{1/2} \geqq A^2B_2 - 2V_A\phi AB_2{}^{1/2} \tag{C8}$$

It is convenient to add $2V_A\phi AB_1{}^{1/2} - A^2B_2$ to both sides. This allows inequality (C8) to be rewritten as

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$$A^2\left(B_1{}^{1/2} + B_2{}^{1/2}\right)\left(B_1{}^{1/2} - B_2{}^{1/2}\right) \geqq 2V_A\phi A\left(B_1{}^{1/2} - B_2{}^{1/2}\right) \tag{C8a}$$

By definition, $B = \dfrac{1 - \cos\theta}{1 - \frac{1}{\phi}\cos\theta}$; therefore,

$$B_1{}^{1/2} = \left(\frac{2}{1 + \frac{1}{\phi}}\right)^{1/2} \tag{C9}$$

For condition 2 equation (C6) yields

$$-1 \leqq \frac{1 - \left(\frac{A}{V_A\phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A\phi}\right)^2} \tag{C10}$$

By introducing the value of $\left(\frac{A}{V_A\phi}\right)^2$ from equation (C5a), inequality (C10) simplifies to

$$B_2 \geqq \frac{2}{1 + \frac{1}{\phi}} \tag{C11a}$$

and

$$B_2^{1/2} \geqq \left(\frac{2}{1 + \frac{1}{\phi}}\right)^{1/2} \tag{C11b}$$

From equation (C9) and inequality (C11b) it follows that

$$B_1^{1/2} - B_2^{1/2} \leqq 0 \tag{C12}$$

Inequality (C8a), which is the condition for $V_{R1}^2 \geqq V_{R2}^2$ which in turn implies that V_{R2} is a minimum, is now divided by $\left(B_1^{1/2} - B_2^{1/2}\right)A^2$, a negative quantity which reverses the inequality, and hence

$$B_1^{1/2} + B_2^{1/2} \leqq \frac{2V_A\phi}{A} \tag{C13}$$

From equation (C5a), both sides of which are positive, it is seen that

$$\frac{2V_A\phi}{A} = 2B_2^{1/2} \tag{C14}$$

Therefore, substitution of equation (C14) into inequality (C13) yields

$$B_1^{1/2} + B_2^{1/2} \leqq 2B_2^{1/2} \tag{C15}$$

Subtraction of $2B_2^{1/2}$ from both sides of inequality (C15) yields

$$B_1^{1/2} - B_2^{1/2} \leqq 0 \tag{C16}$$

but from inequality (C12), it is known that this is true; therefore, when V_{R2} exists, it is a minimum. It must be noted that V_{R2} is a minimum for a conic trajectory. For the trajectory to be elliptical, two-body orbital equations yield equation (15) and for elliptical conditions to exist

$$2 \frac{r_p}{r} - 1 - \cos \theta > 0$$

or

$$2 \frac{r_p}{r} - 1 > \cos \theta \tag{C17}$$

Hence, when V_{R2} exists and $2 \frac{r_p}{r} - 1 > \cos \theta$, then V_{R2} is the minimum retrograde velocity for an elliptical orbit.

In summation, if V_{R2} does exist, it is a minimum and, of necessity,

$$\frac{\partial V_R{}^2}{\partial \theta} = 0 \quad \text{and} \quad -1 \leqq \frac{1 - \left(\frac{A}{V_A \phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A \phi}\right)^2} \leqq 1.$$

When $\phi = 1$, $\theta = 0$ is the unique value which satisfies equation (C4).

If V_{R2} does not exist and $\phi \neq 1$, then V_{R1} is a minimum. If V_{R1} is a minimum, $\theta = 180^\circ$ and, of necessity, $\frac{\partial V_R{}^2}{\partial \theta} = 0$; also implied is that $2 \frac{r_p}{r} - 1 > \cos \theta$, which is the condition for ellipticity.

The method for determining the minimum retrograde velocity for a given set of conditions is outlined as follows:

Step 1:

Determine from the following equation if V_{R2} does exist:

$$\theta = \cos^{-1}\left[\frac{1 - \left(\frac{A}{V_A\phi}\right)^2}{\frac{1}{\phi} - \left(\frac{A}{V_A\phi}\right)^2}\right] \tag{C18}$$

If there is a solution to equation (C18), then V_{R2} does exist and the determined value of θ will yield the minimum retrograde impulse. Should the equation have no legitimate solution, then V_{R2} does not exist and, therefore, $\theta = 180^\circ$, which is the value of θ for V_{R1}, will yield the minimum retrograde impulse.

Step 2:

The eccentricity of the orbit should now be determined. By using the value of θ as found from equation (C18), the eccentricity can be determined from

$$\epsilon = \frac{B}{\phi} - 1 \tag{C19}$$

Step 3:

The minimum retrograde velocity for an elliptical orbit can be ascertained by using the determined value of θ in equation (20) which is

$$V_R^2 = V_A^2 + A^2B + K - 2V_A\phi A B^{1/2}$$

If the resultant orbit is not elliptical, the minimum retrograde velocity can be determined from equation (23) which is

$$V_R^2 = V_A^2 + A^2B + K + 2V_A\phi A B^{1/2}$$

REFERENCE

1. Buchheim, R. W.: Lunar Flight Trajectories. P-1268, The RAND Corp., Jan. 30, 1958.

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TABLE 1

ORBIT REFERENCE

	$\theta = 120^\circ$	$\theta = 150^\circ$	$\theta = 180^\circ$
Arrival altitude, 1,000 nautical miles			
For h_p = 50 nautical miles,			
h_a, nautical miles	1,914.0	1,143.6	1,000.0
V_{oa}, ft/sec	2,263.8	2,953.8	3,146.1
V_{op}, ft/sec	6,535.5	6,244.9	6,171.7
Period, hr	5.3	3.8	3.5
For h_p = 100 nautical miles,			
h_a, nautical miles	1,788.0	1,128.6	1,000.0
V_{oa}, ft/sec	2,397.1	3,031.5	3,197.5
V_{op}, ft/sec	6,296.8	6,036.9	5,970.9
Period, hr	5.1	3.8	3.6
For h_p = 200 nautical miles,			
h_a, nautical miles	1,593.0	1,102.9	1,000.0
V_{oa}, ft/sec	2,638.5	3,156.7	3,293.2
V_{op}, ft/sec	5,868.9	5,661.8	5,608.7
Period, hr	4.9	4.0	3.8
Arrival altitude, 3,000 nautical miles			
For h_p = 50 nautical miles,			
h_a, nautical miles	[a]844,339.7	4,074.3	3,000.0
V_{oa}, ft/sec	8.8	1,356.0	1,701.0
V_{op}, ft/sec	7,578.9	6,931.0	6,780.4
Period, hr	17,330.2	10.3	7.7
For h_p = 100 nautical miles,			
h_a, nautical miles	[a]56,562.3	3,938.0	3,000.0
V_{oa}, ft/sec	132.3	1,416.6	1,734.6
V_{op}, ft/sec	7,333.1	6,724.4	6,582.3
Period, hr	315.3	10.2	7.8
For h_p = 200 nautical miles,			
h_a, nautical miles	[a]20,965.9	3,845.1	3,000.0
V_{oa}, ft/sec	357.9	1,510.8	1,798.3
V_{op}, ft/sec	6,889.2	6,350.8	6,223.8
Period, hr	77.9	10.1	8.0
Arrival altitude, 5,000 nautical miles			
For h_p = 50 nautical miles,			
h_a, nautical miles	Hyperbolic	8,335.5	5,000.0
V_{oa}, ft/sec	Trajectory	757.9	1,168.2
V_{op}, ft/sec		7,209.1	7,021.7
Period, hr		23.1	12.8
For h_p = 100 nautical miles,			
h_a, nautical miles	Hyperbolic	8,044.8	5,000.0
V_{oa}, ft/sec	Trajectory	809.3	1,193.0
V_{op}, ft/sec		7,005.4	6,826.5
Period, hr		22.3	13.0
For h_p = 200 nautical miles,			
h_a, nautical miles	Hyperbolic	7,578.9	5,000.0
V_{oa}, ft/sec	Trajectory	836.6	1,240.4
V_{op}, ft/sec		6,636.5	6,472.8
Period, hr		21.1	13.2

[a]In the solar system the assumption of elliptical charecteristics in these cases is not valid.

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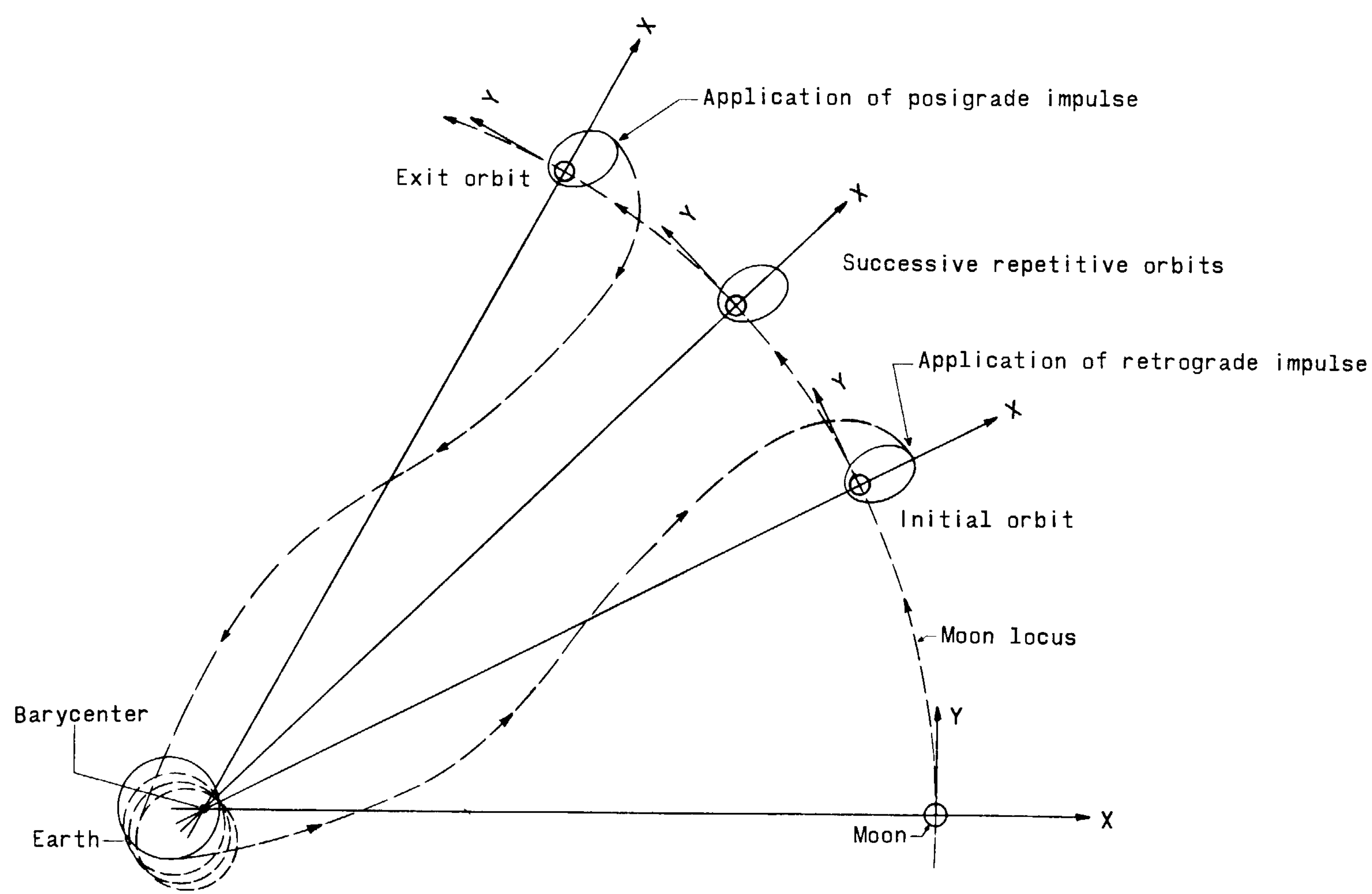

Figure 1.- Diagram of envisaged manned lunar mission.

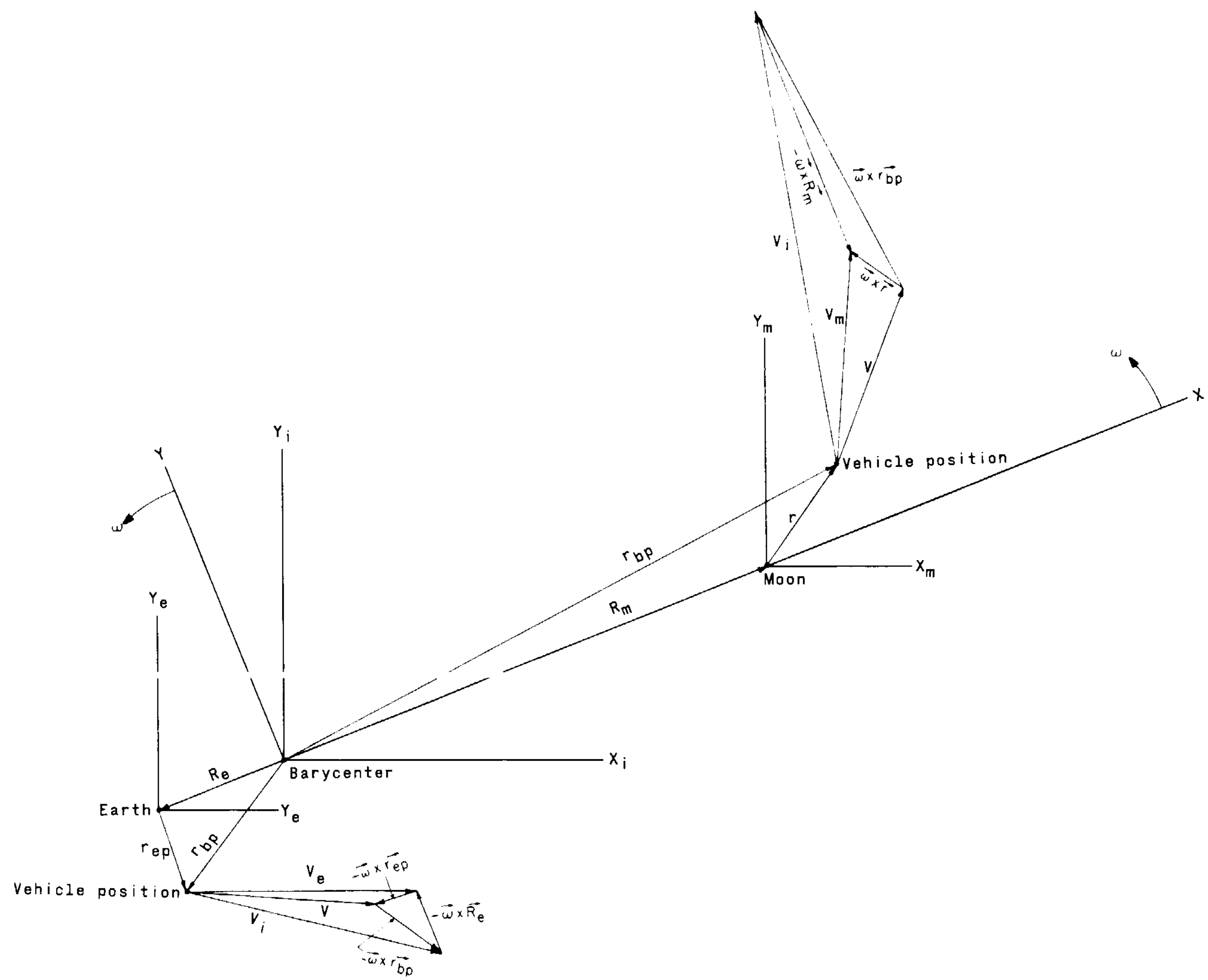

Figure 2.- The relationship of velocity vectors. In this diagram all vectors are constrained to the earth-moon plane.

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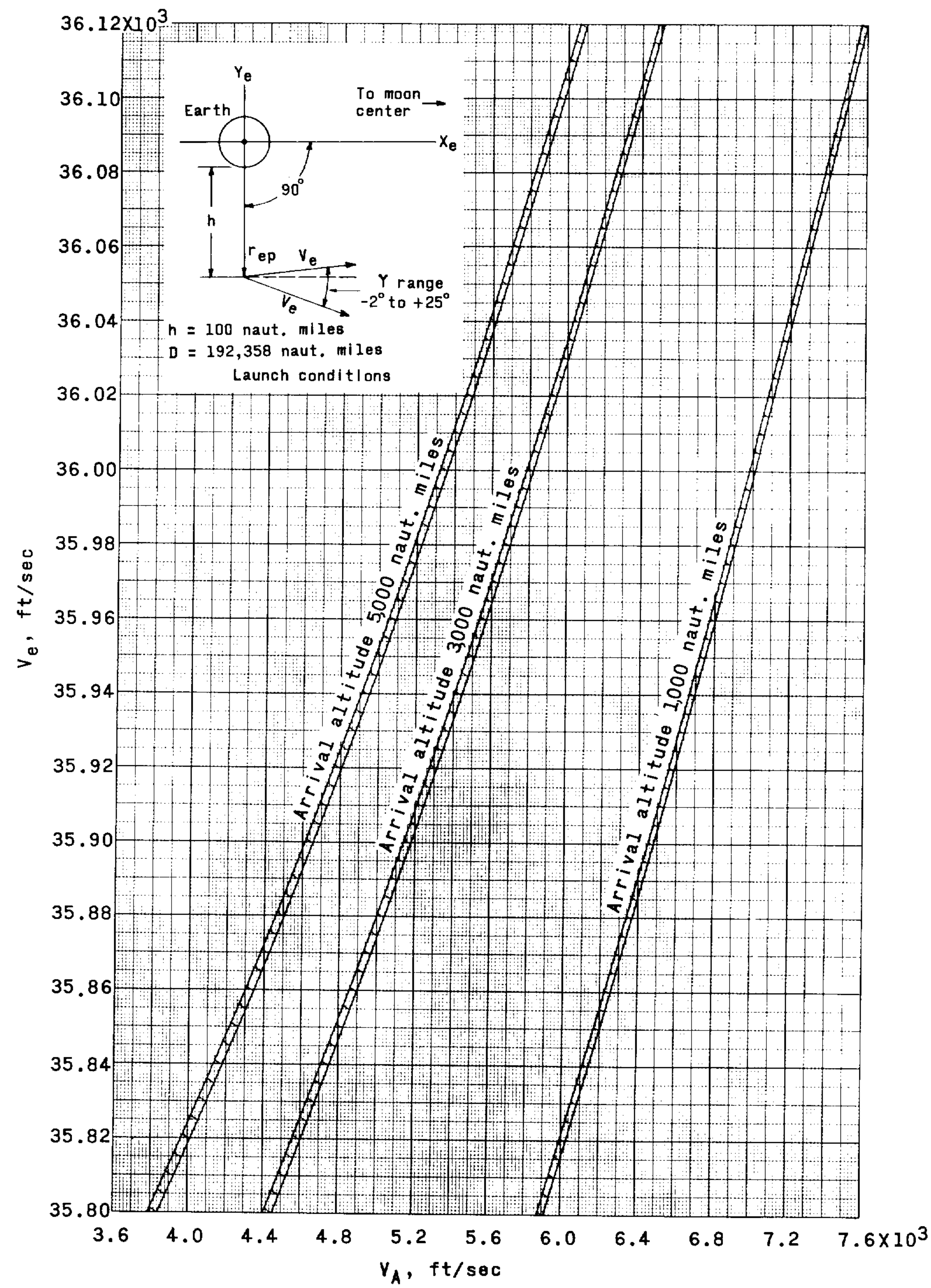
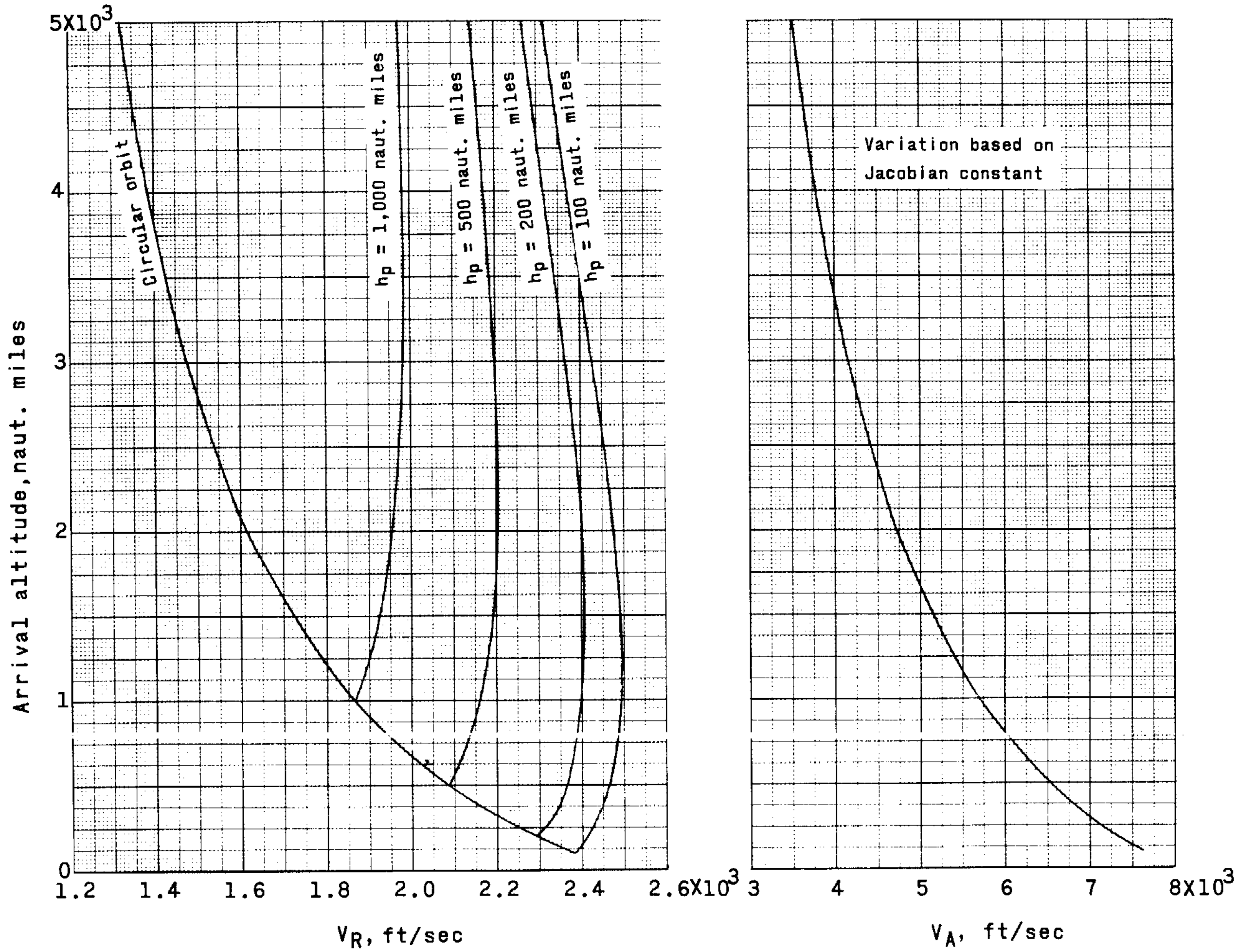


Figure 3.- Variation of earth departure velocity V_e with lunar arrival velocity V_A as determined by Jacobian relationship.



(a) $V = 35{,}707$ ft/sec.

Figure 6.- Retrograde velocity for insertion into orbit for $\theta = 180^{\circ}$. Earth departure conditions; altitude, 100 nautical miles; moon lead angle, 90°; earth-moon distance, 192,358 nautical miles.

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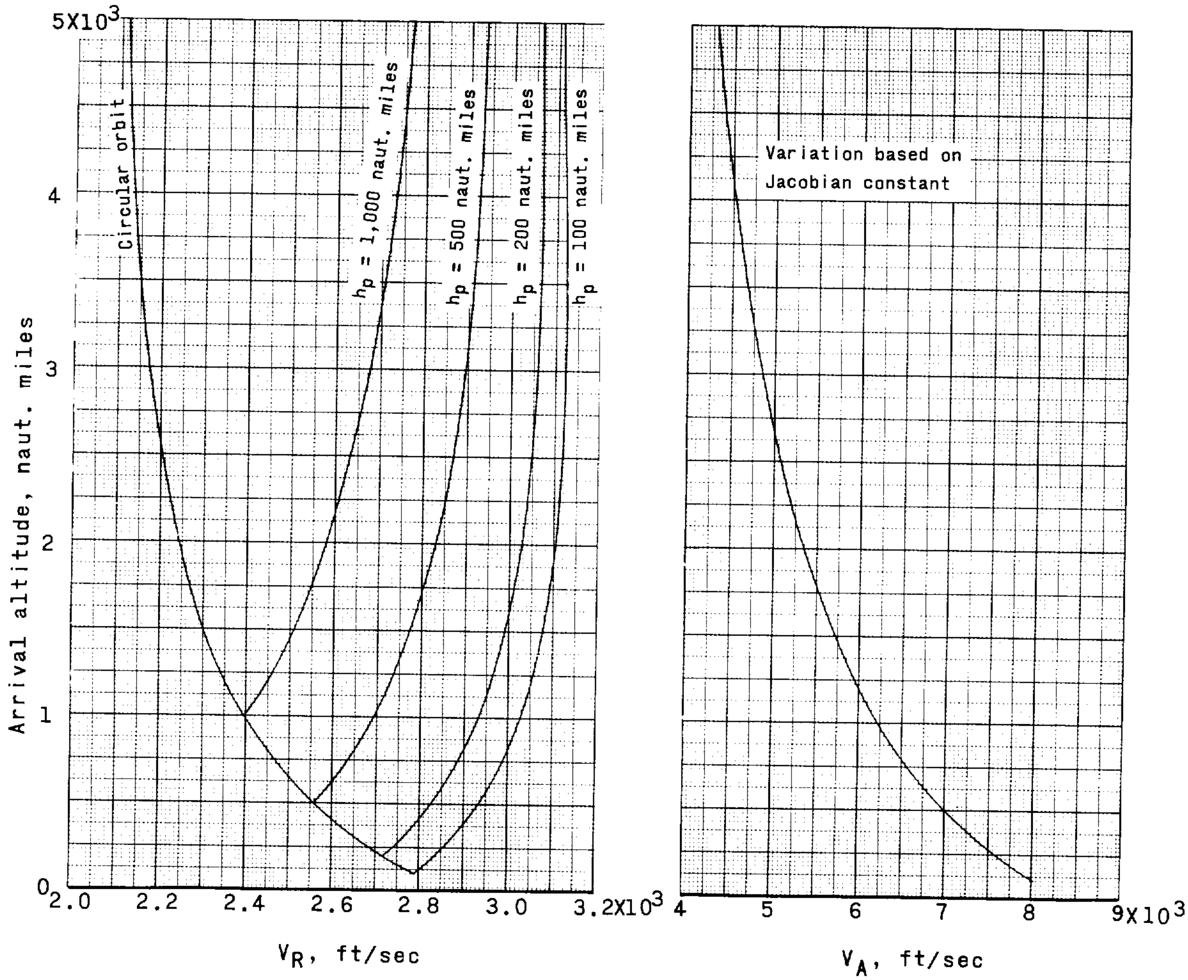
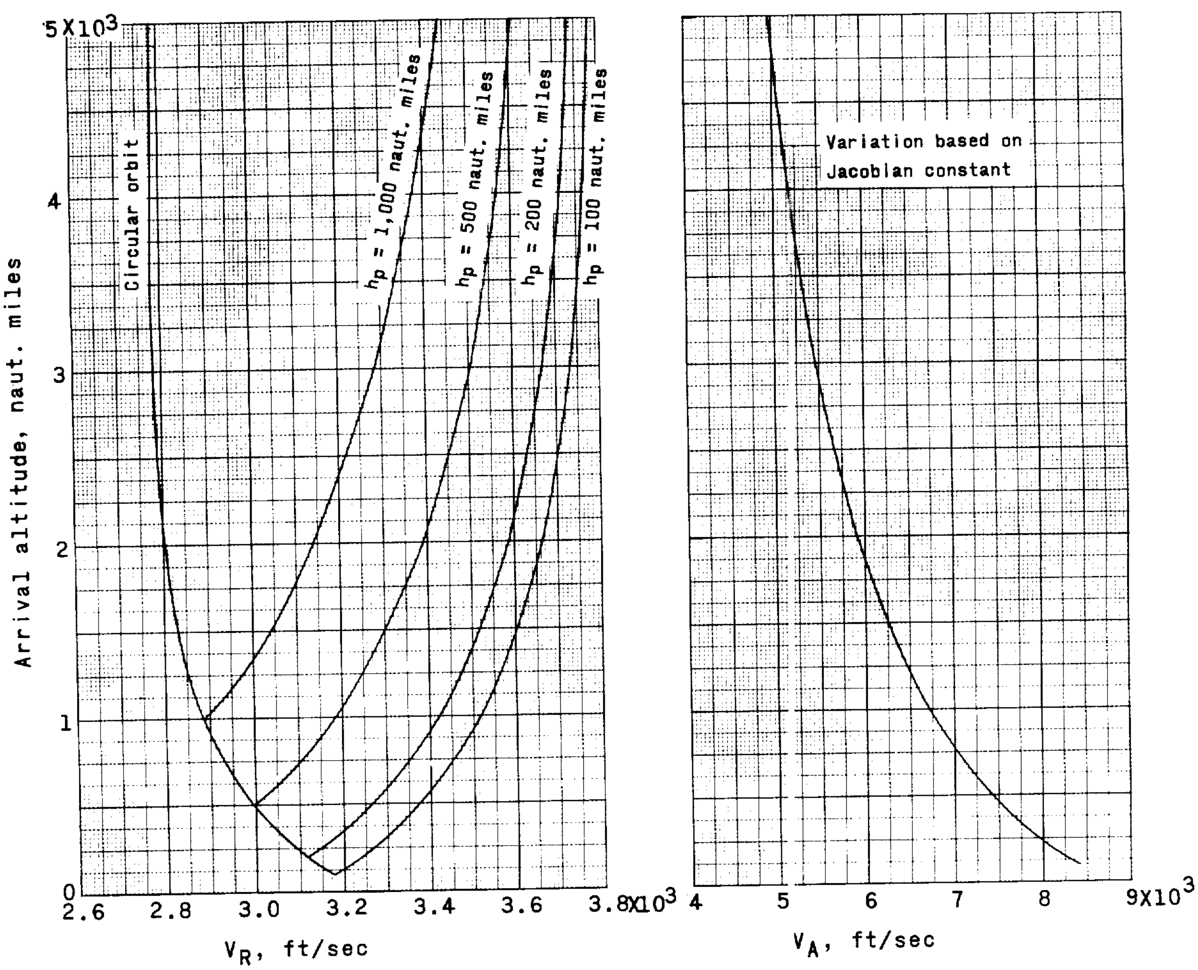


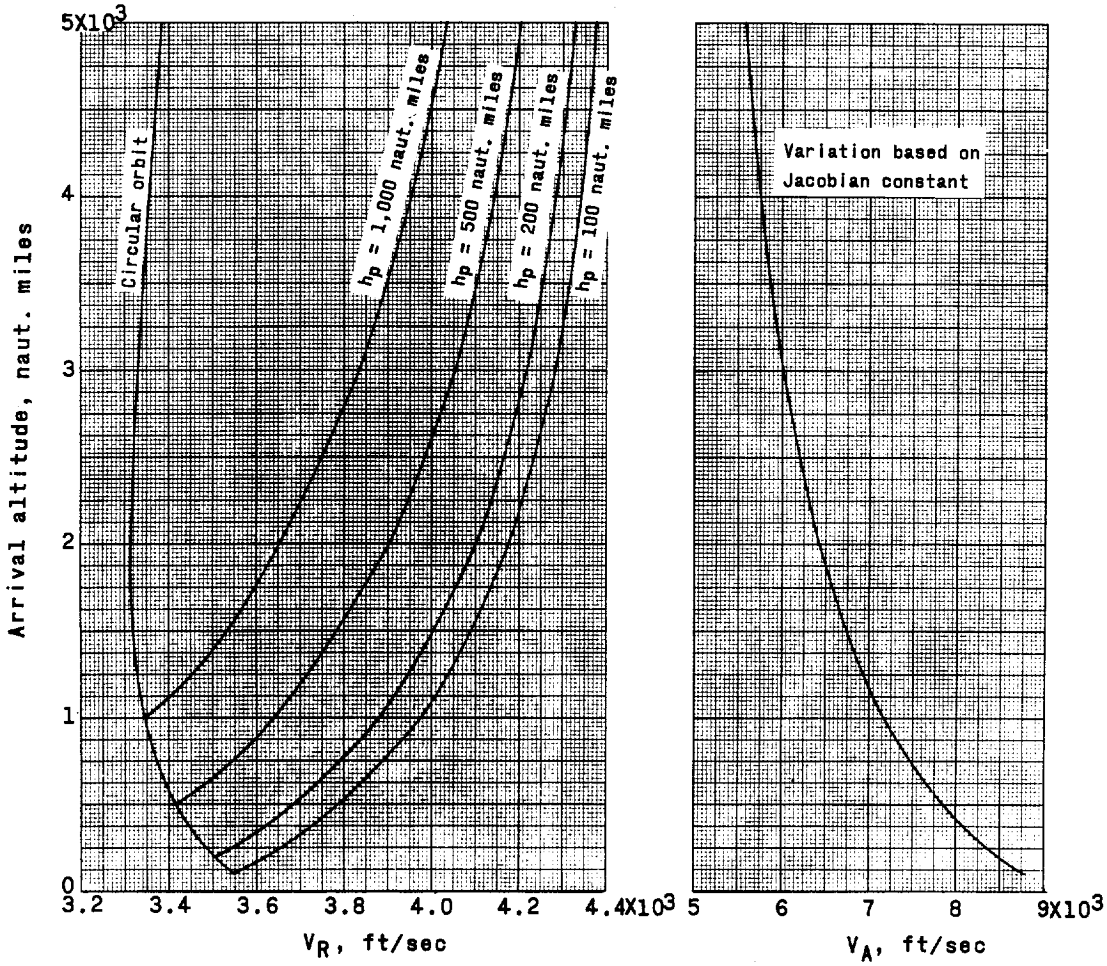
(b) V = 35,796 ft/sec.

Figure 6.- Continued.



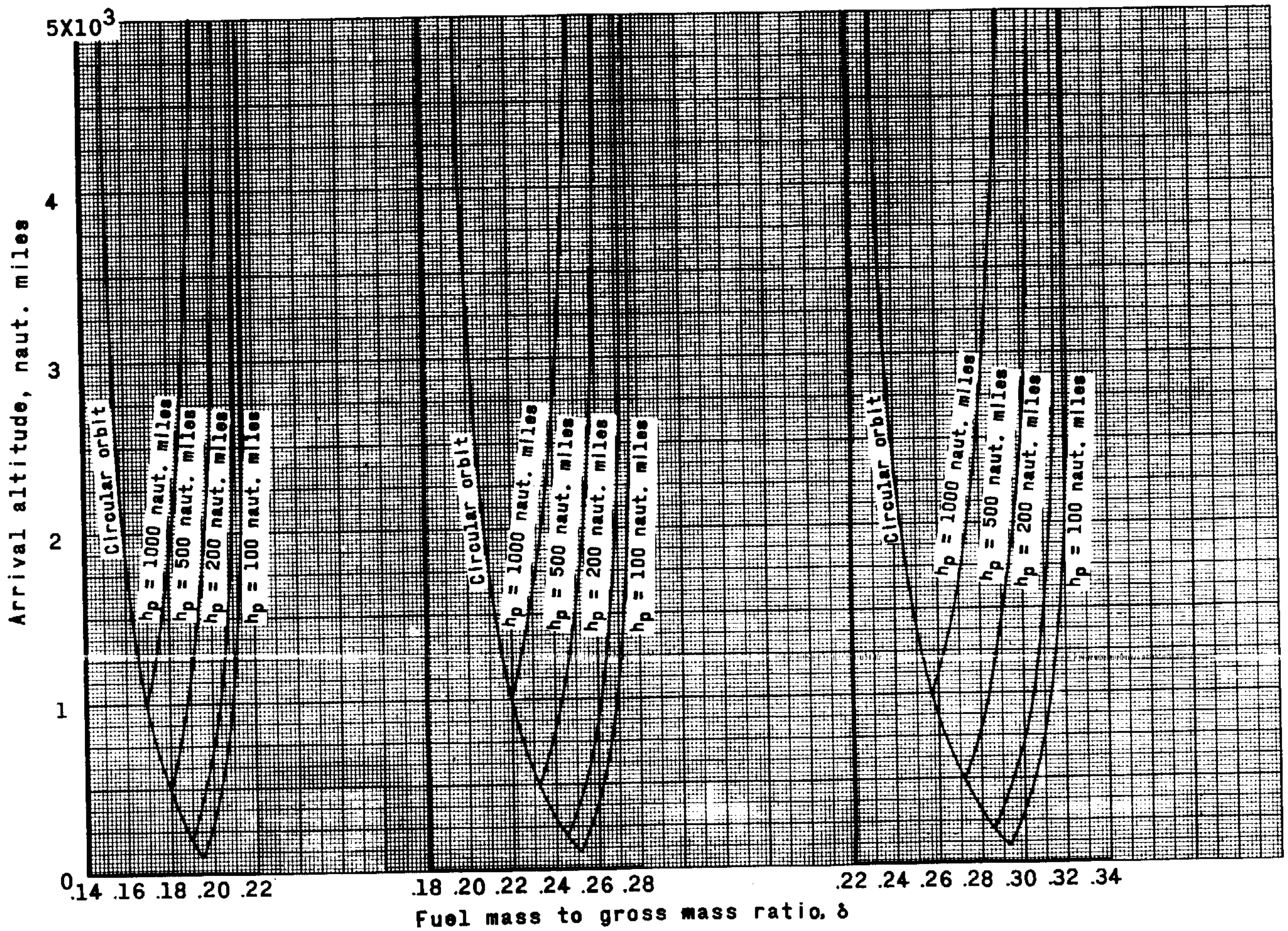
(c) V = 35,885 ft/sec.

Figure 6.- Continued.



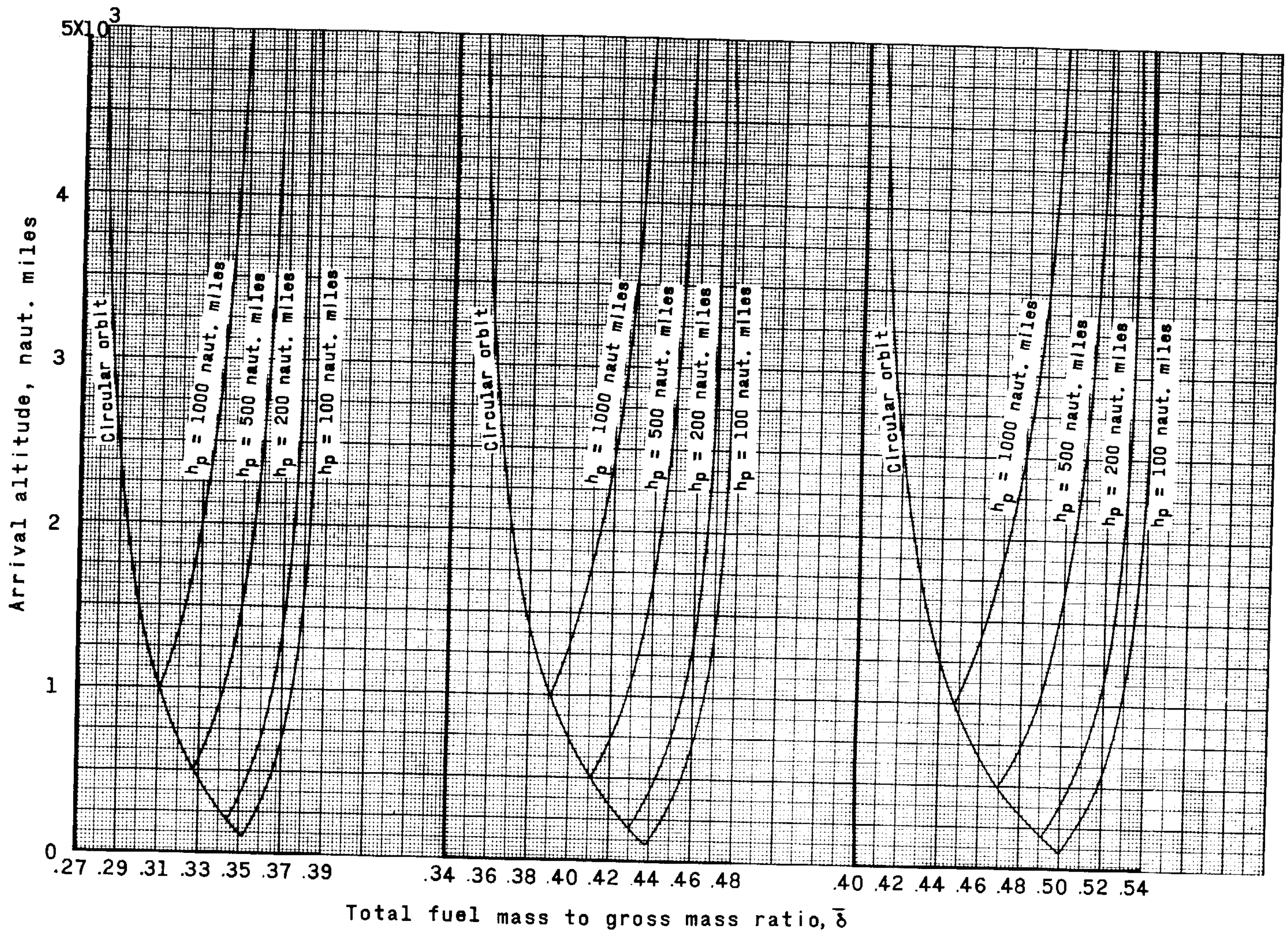
(d) $\dot{V}$ = 35,974 ft/sec.

Figure 6.- Concluded.



(a) I = 400 sec. (b) I = 300 sec. (c) I = 250 sec.

Figure 7.- Fuel consumption for insertion into orbit.



(a) I = 400 sec. (b) I = 300 sec. (c) I = 250 sec.

Figure 8.- Total fuel consumption for orbit entry and exit.